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$*$-GENERALIZED POLYNOMIAL IDENTITIES OF FINITE DIMENSIONAL CENTRAL SIMPLE ALGEBRAS[†]

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ABSTRACT
Let D be a finite dimensional central simple algebra with involution $*$ of the first kind. We prove (in a fixed number of variables) the ideal of $*$-GPI's of D is generated by a finite collection of elements of the form $[v_{ij}, v_{pq}]$ and $v_{ij} - v_{ij}^*$ where the v_{ij}'s are first degree generalized polynomials.

Much of the work in this article was influenced by a result of Procesi [2], [3].

Let D denote a finite dimensional central simple algebra with involution $*$ of the first kind over an infinite field K. Let X be the set consisting of the $2m$ noncommuting indeterminates $x_1, \cdots, x_m, z_1, \cdots, z_m$. Form the ring $R = D_K\langle X\rangle$ which is the free product of D with the free noncommutative K-algebra $K\langle X\rangle$. The elements of R can be written as $\sum \alpha_J d_{i_0} y_{j_1} d_{i_1} \cdots y_{j_{2m}} d_{i_{2m}}$ where $y_{j_k} \in X$, $\alpha_J \in K$, and $d_{i_p} \in D$. The involution $*$ on D may be extended to an involution on R, also denoted $*$, as follows: $d \to d^*$ for $d \in D$, $x_i \to z_i$ and $z_i \to x_i$, $i = 1, \cdots, m$.

Set $S = \{(d_1, \cdots, d_m, d_1^*, \cdots, d_m^*) \mid d_i \in D\}$.

DEFINITION. $f(x_1, \cdots, x_m, z_1, \cdots, z_m) \in R$ is a *$*$-generalized polynomial identity* ($*$-GPI) of D if $f(p) = 0$ for all $p \in S$.

Let D^S denote the set of all functions from S into D. D^S is made into a ring under pointwise operations. Any $f \in R$ induces a function in D^S as follows: define $\phi_f : S \to D$ by

$$(d_1, \cdots, d_m, d_1^*, \cdots, d_m^*) \to f(d_1, \cdots, d_m, d_1^*, \cdots, d_m^*).$$

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This gives rise to a homomorphism $\phi : R \to D^S$, namely $f \to \phi_f$. Ker ϕ is equal to the ideal of $*$-GPI's of D. We will show Ker ϕ is finitely generated and determine a collection of generators.

If $a \in D$, let a_r and a_l denote respectively the right and left multiplications determined by a. a_r and a_l belong to the ring $\mathrm{End}_K D$ of all K-linear transformations on D. We make $\mathrm{End}_K D$ into a D-bimodule via $\sigma \cdot a = \sigma a_r$ and $a \cdot \sigma = \sigma a_l$ for all $\sigma \in \mathrm{End}_K D$ and $a \in D$ (applying maps on the right).

DEFINITION. The *opposite ring* D^0 of D is the ring having the same additive structure as D but multiplication in reverse order (i.e., the product of a and b in D^0 is ba).

We make D^0 into a D-bimodule by giving it the same structure as D. The mapping $D^0 \otimes_K D \to \mathrm{End}_K D$ given by $\Sigma\, a_i \otimes b_i \to \Sigma\, a_{i_l} b_{i_r}$ is a ring homomorphism as well as a D-bimodule mapping. Since $D^0 \otimes_K D$ is simple (D being central simple over K), this map is one-to-one. Checking dimensions over K, we see that it is onto and hence a D-bimodule isomorphism. We now have the following sequence of D-bimodule isomorphisms:

$$D \otimes_K D \to D^0 \otimes_K D \to \mathrm{End}_K D, \tag{1}$$

$$\sum a_i \otimes b_i \to \sum a_i \otimes b_i \to \sum a_{i_l} b_{i_r}.$$

The following maps are easily seen to be D-bimodule isomorphisms:

$$D \otimes_K D \to D x_i D, \tag{2}$$

$$\sum_j a_j \otimes b_j \to \sum_j a_j x_i b_j, \qquad i = 1, \cdots, m;$$

$$D \otimes_K D \to D z_i D, \tag{3}$$

$$\sum_j a_j \otimes b_j \to \sum_j a_j z_i b_j, \qquad i = 1, \cdots, m.$$

Combining (1), (2), and (3), we obtain the following D-bimodule isomorphisms:

$$D x_i D \to D^0 \otimes_K D \to \mathrm{End}_K D, \tag{4}$$

$$\sum_j a_j x_i b_j \to \sum_j a_j \otimes b_j \to \sum_j a_{j_l} b_{j_r},$$

$$D z_i D \to D^0 \otimes_K D \to \mathrm{End}_K D, \tag{5}$$

$$\sum_j a_j z_i b_j \to \sum_j a_j \otimes b_j \to \sum_j a_{j_l} b_{j_r}.$$

Suppose $\{u_1, \cdots, u_n\}$ is a K-basis of D. Write $D^{\#}$ for the dual space $\mathrm{Hom}_K(D, K) \subset \mathrm{End}_K D$. Let $\{u_1^{\#}, \cdots, u_n^{\#}\}$ be the corresponding dual basis of $D^{\#}$. $\{u_1^*, \cdots, u_n^*\}$ is also a K-basis for D and let $\{(u_1^*)^{\#}, \cdots, (u_n^*)^{\#}\}$ be the corresponding dual basis. If $d \in D$, then $d = \sum_{k=1}^n \alpha_k u_k^*$ for $\alpha_k \in K$. Therefore $d(u_j^*)^{\#} = \alpha_j = d^* u_j^{\#}$. Hence $(u_j^*)^{\#} = *u_j^{\#}$, $j = 1, \cdots, n$.

Since $u_j^{\#}$, $(u_j^*)^{\#} \in \mathrm{End}_K D$, then (1) implies

$$u_j^{\#} = \sum_{\alpha=1}^{n} (a_{j\alpha})_l (u_\alpha)_r \tag{6}$$

and

$$(u_j^*)^{\#} = \sum_{\alpha=1}^{n} (u_\alpha^*)_l (b_{j\alpha})_r \tag{7}$$

for $a_{j\alpha}$, $b_{j\alpha} \in D$ and $j = 1, \cdots, n$. By (4) and (5), we have:

$$\sum_{\alpha=1}^{n} a_{j\alpha} x_i u_\alpha \to \sum_{\alpha=1}^{n} a_{j\alpha} \otimes u_\alpha \to \sum_{\alpha=1}^{n} (a_{j\alpha})_l (u_\alpha)_r = u_j^{\#}, \tag{8}$$

$$\sum_{\alpha=1}^{n} u_\alpha^* z_i b_{j\alpha} \to \sum_{\alpha=1}^{n} u_\alpha^* \otimes b_{j\alpha} \to \sum_{\alpha=1}^{n} (u_\alpha^*)_l (b_{j\alpha})_r = (u_j^*)^{\#}, \tag{9}$$

where $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$.

LEMMA. 1. $b_{j\alpha} = a_{j\alpha}^*$.

PROOF. $(u_j^*)^{\#} = * u_j^{\#}$ implies that for all $d \in D$, $d^*(u_j^*)^{\#} = du_j^{\#}$. Using (6) and (7), this can be written as

$$\sum_{\alpha=1}^{n} u_\alpha^* d^* b_{j\alpha} = \sum_{\alpha=1}^{n} a_{j\alpha} d u_\alpha. \tag{A}$$

Since $(u_j^*)^{\#} \in D^{\#}$, $d^*(u_j^*)^{\#} = \sum_{\alpha=1}^n u_\alpha^* d^* b_{j\alpha} \in K$ for all $d \in D$. $*$ an involution of the first kind implies

$$\sum_{\alpha=1}^{n} u_\alpha^* d^* b_{j\alpha} = \left(\sum_{\alpha=1}^{n} u_\alpha^* d^* b_{j\alpha} \right)^* = \sum_{\alpha=1}^{n} b_{j\alpha}^* d u_\alpha.$$

From (A), we obtain:

$$\sum_{\alpha=1}^{n} a_{j\alpha} d u_\alpha = \sum_{\alpha=1}^{n} b_{j\alpha}^* d u_\alpha \qquad \text{for all } d \in D.$$

Hence $\sum_{\alpha=1}^n (a_{j\alpha} - b_{j\alpha}^*)_l (u_\alpha)_r = 0$. Applying (1), we have

$$\sum_{\alpha=1}^{n} ((a_{j\alpha} - b_{j\alpha}^*) \otimes u_\alpha) = 0.$$

By the independence of $\{u_1, \cdots, u_n\}$, we obtain $b_{j\alpha} = a^*_{j\alpha}$. □

Set $v_{ij} = \Sigma^n_{\alpha=1} a_{j\alpha} x_i u_\alpha$ and $w_{ij} = \Sigma^n_{\alpha=1} u^*_\alpha z_i a^*_{j\alpha}$. Note that $w_{ij} = v^*_{ij}$, where $*$ is the involution on R.

Lemma 2. *The v_{ij}'s and w_{ij}'s commute with D.*

Proof. For each $i = 1, \cdots, m$, let $\psi_i : Dx_iD \to \mathrm{End}_K D$ be the map defined in (4). ψ_i is a D-bimodule isomorphism and $u^\#_j = v_{ij}\psi_i$. For any $a \in D$, $a(u^\#_j d_l) = d(au^\#_j) = (au^\#_j)d$ (since $au^\#_j \in K) = a(u^\#_j d_r)$. Hence $d \cdot u^\#_j = u^\#_j \cdot d$. For any $d \in D$, $(dv_{ij})\psi_i = d \cdot (v_{ij}\psi_i) = d \cdot u^\#_j = u^\#_j \cdot d = (v_{ij}\psi_i) \cdot d = (v_{ij}d)\psi_i$. Therefore $dv_{ij} = v_{ij}d$.

Now $dw_{ij} = dv^*_{ij} = (v_{ij}d^*)^* = (d^*v_{ij})^* = v^*_{ij}d = w_{ij}d$. Hence the v_{ij}'s and w_{ij}'s commute with D. □

Lemma 3. *For $i = 1, \cdots, m$, $Dx_iD = \Sigma^n_{j=1} v_{ij}D$ and $Dz_iD = \Sigma^n_{j=1} w_{ij}D$.*

Proof. Since $v_{ij} \in Dx_iD$, the inclusion $\Sigma^n_{j=1} v_{ij}D \subseteq Dx_iD$ is clear. By (4), $Dx_iD \cong \mathrm{End}_K D$ (as D-bimodules). $\mathrm{End}_K D$ is (freely) generated as a D-bimodule by the $u^\#_j$'s; hence Dx_iD is (freely) generated as a D-bimodule by the set $\{v_{ij} \mid 1 \leqq j \leqq n\}$. In particular, $x_i = \Sigma^n_{j=1} v_{ij}d_j$ for $d_j \in D$. Thus for any a, $b \in D$, $ax_ib = \Sigma^n_{j=1} av_{ij}d_jb = \Sigma^n_{j=1} v_{ij}ad_jb$. Hence $Dx_iD \subseteq \Sigma^n_{j=1} v_{ij}D$ and we have equality.

Now $Dz_iD = Dx^*_iD = (\Sigma^n_{j=1} v_{ij}D)^* = \Sigma^n_{j=1} Dw_{ij} = \Sigma^n_{j=1} w_{ij}D$. □

Let $D\langle v_{ij}, w_{ij}\rangle$ be the subring of R generated by D, the v_{ij}'s, and the w_{ij}'s. Lemma 3 implies $R = D\langle v_{ij}, w_{ij}\rangle$. So $\phi : D\langle v_{ij}, w_{ij}\rangle \to D^S$. Set $\xi_{ij} = v_{ij}\phi$ and $\zeta_{ij} = w_{ij}\phi$. Now $0 = 0\phi = [d, v_{ij}]\phi = [d, \xi_{ij}]$. Hence ξ_{ij} maps S into K. Therefore, $\xi_{ij} = \xi^*_{ij} = v^*_{ij}\phi = w_{ij}\phi = \zeta_{ij}$. It follows that the image of ϕ must be generated as a ring by D and ξ_{ij}.

Lemma 4. *The ξ_{ij}'s are algebraically independent.*

Proof. Let $\{t_{ij} \mid 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$ be a collection of commuting indeterminates. Suppose $f(t_{11}, \cdots, t_{mn}) \in D[t_{11}, \cdots, t_{mn}]$ (the polynomial ring in the t_{ij}'s with coefficients from D) and $f(\xi_{11}, \cdots, \xi_{mn}) = 0$. Choose $\beta_{11}, \cdots, \beta_{mn} \in K$ and for each $k = 1, \cdots, m$, set $d_k = \Sigma^n_{\lambda=1} \beta_{k\lambda}u_\lambda$ and let $p = (d_1, \cdots, d_m, d^*_1, \cdots, d^*_m) \in S$. Now $u^\#_j = \Sigma^n_{\alpha=1} (a_{j\alpha})_l (u_\alpha)_r$ implies

$$\beta_{ij} = \left(\sum_{\lambda=1}^n \beta_{i\lambda}u_\lambda \right) u^\#_j = \sum_{\alpha=1}^n a_{j\alpha} \left(\sum_{\lambda=1}^n \beta_{i\lambda}u_\lambda \right) u_\alpha = \sum_{\alpha=1}^n a_{j\alpha}d_iu_\alpha = p\xi_{ij}.$$

Thus $0 = pf(\xi_{11}, \cdots, \xi_{mn}) = f(\beta_{11}, \cdots, \beta_{mn})$. So f vanishes on all substitutions from K, the infinite center of D. By a Vandermonde argument (e.g., see [4]) we may conclude $f = 0$. This proves that the ξ_{ij}'s are algebraically independent. □

Lemma 4 together with the remarks preceding it imply ϕ maps R onto the polynomial ring $D[\xi_{ij}]$.

If a and b are elements of a ring, let $[a, b] = ab - ba$. Since ξ_{ij} maps into K, $[\xi_{ij}, \xi_{pq}] = 0$. We also have $\xi_{ij} - \zeta_{ij} = 0$. Hence $[v_{ij}, v_{pq}]$ and $v_{ij} - w_{ij}$ are in Ker ϕ. Let I denote the ideal of R generated by the set

$$\{[v_{ij}, v_{pq}], v_{ij} - w_{ij} \mid 1 \leqq i, p \leqq m, 1 \leqq j, q \leqq n\}.$$

LEMMA 5. $I = \operatorname{Ker} \phi$.

PROOF. The above remark implies $I \subseteq \operatorname{Ker} \phi$. Suppose $f \in \operatorname{Ker} \phi$. Consider the following relations:

$$v_{ij}v_{pq} = [v_{ij}, v_{pq}] + v_{pq}v_{ij},$$

$$v_{ij}w_{pq} = v_{ij}(w_{pq} - v_{pq}) + [v_{ij}, v_{pq}] + v_{pq}v_{ij},$$

$$w_{ij}v_{pq} = (w_{ij} - v_{ij})v_{pq} + [v_{ij}, v_{pq}] + v_{pq}v_{ij},$$

$$w_{ij}w_{pq} = (v_{ij} - w_{ij})(v_{pq} - w_{pq}) + v_{ij}w_{pq} + w_{ij}v_{pq} - v_{ij}v_{pq}.$$

These relations allow us to write $f = g + h$, where $g \in I$ and h is of the form $h = \sum d_J v_{11}^{l_{11}} v_{12}^{l_{12}} \cdots v_{mn}^{l_{mn}}$, $d_J \in D$.

Applying ϕ, we have

$$0 = f\phi = h\phi = \sum d_J \xi_{11}^{l_{11}} \xi_{12}^{l_{12}} \cdots \xi_{mn}^{l_{mn}}.$$

By Lemma 4, each $d_J = 0$. Hence $h = 0$ and $f = g \in I$, proving $\operatorname{Ker} \phi = I$. □

Since $w_{ij} = v_{ij}^*$ we have

THEOREM. *Let D be a finite dimensional central simple algebra (of dimension n) with involution $*$ of the first kind over an infinite field K. Then the ideal (of $R = D_K\langle x_1, \cdots, x_m, z_1, \cdots, z_m\rangle$) of $*$-GPI's of D is finitely generated. The generators are of the form $[v_{ij}, v_{pq}]$, $v_{ij} - v_{ij}^*$ $(1 \leqq i \leqq m, 1 \leqq j \leqq n)$ where the v_{ij}'s are first degree generalized polynomials and v_{ij}^* is the image of v_{ij} under $*$ when extended to R.*

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